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ON OPTIMAL DEPLETION OF EXHAUSTIBLE RESOURCES: EXISTENCE AND CHARACTERIZATION RESULTS[1]

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1. INTRODUCTION

IN RECENT YEARS, there has developed a significant literature on the economics of natural resources. A part of this literature is concerned with finding the characteristics of optimal programs for economies with exhaustible resource constraints; that is, of jointly determining the optimal depletion of such resources, and optimal investment in augmentable capital goods.

It should be noted that the theory of optimal economic growth, in the form given it by Ramsey [**17**], and extensively developed by many others, had been primarily concerned with the latter problem. Natural resources were often assumed to be supplied exogenously in given amounts in each period, an approach clearly unsuitable for capturing the essence of problems associated with the optimal use of exhaustible resources. The more recent literature derives its inspiration from Hotelling's classic paper [**8**], stressing the increasing significance of such resources in production, as they are irrevocably run down.

I shall consider, in this paper, a model of intertemporal allocation in which there is a produced good (which can be used for consumption or for further production), and an exhaustible resource (which is essential for production), the total initial stock of which is given. The use of the resource over the (infinite) planning horizon must not exceed this available stock. A planner is assumed to evaluate consumption in each period, in terms of a utility function, and to "maximize" the undiscounted sum of these one-period utilities, to obtain, simultaneously, the optimal depletion of the exhaustible resource, and the optimal investment pattern.[2] This model resembles, in some aspect or other, the frameworks examined by Dasgupta [**3**], Dasgupta and Heal [**4**, **5**], Solow [**18**], and Stiglitz [**19**], to mention only a few.

I will address three sets of issues in this framework. First, an interesting problem in the theory of optimal economic growth is to find suitable conditions under which a competitive program is optimal. I will show that a feasible program is optimal if and only if (a) it is competitive,[3] and (b) it satisfies the transversality condition, that the value of the capital and resource stocks, at the competitive prices, converges to zero (Theorem 3.1). It follows from this result, that a competitive program is optimal if and only if it satisfies the above-mentioned transversality condition. It should be noted that in traditional models of optimal growth (in which exhaustible resources do not appear as essential factors of production), a feasible program, under certain technological curvature conditions, is shown to be optimal if and only if it is competitive, and the value of input

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[2] For the precise sense in which a program is called optimal, see Section 2.

[3] For a definition of this concept, see Section 2.

stocks at the competitive prices is bounded above.[4] Thus, the introduction of the exhaustible resource leads to the following qualitative difference in the characterization of optimality: competitive optimality is signaled by the transversality condition, rather than the input value boundedness condition, holding.

Second, I study the asymptotic properties of optimal programs in this model. The most interesting of these properties are that (a) the consumption level increases monotonically to infinity, and (b) the relative price of the consumption good to the exhaustible resource decreases monotonically to zero (Theorem 4.1). The first result shows that the substitution possibilities of capital for the resource are given sufficient prominence, under undiscounted utility maximization, to make it optimal for consumption levels to grow indefinitely into the future. (It should be noted that when future utilities are discounted, the results of Dasgupta [**3**], and Dasgupta and Heal [**4**], show that the consumption could decrease to zero in the future.) The second result states that, relative to the consumption good, the social valuation placed on the exhaustible resource rises, as its stock is depleted.[5]

Third, I will study the problem of existence of optimal programs in this framework. It should be noted that Solow [**18**] and Dasgupta and Heal [**5**] prove the existence of an optimal program by a method of "construction" which relies heavily on the parametric forms of the production and utility functions, which they use. Since these functions take general forms in this paper, I follow the alternative approach, used widely in proving the existence of optimal programs in traditional growth models (see, especially, Gale [**6**] and Brock [**1**]). This method has three main steps. First, one establishes the existence of a "good" program. Second, any program which is not "good" is shown to be ineligible as a candidate for an optimal program. Third, in the class of good programs, one finds a program which maximizes the sum of undiscounted utilities, and this is shown to be optimal. In contrast to the traditional models, it turns out that the difficult step in the three-part procedure is the first, since one has to ensure that, as the resource stock is (rapidly) depleted, consumption can still grow fast enough to make the utility sum converge. This, in turn, suggests that an interconnecting condition between the production and utility functions will be needed to ensure that a good (and hence an optimal) program exists.

To formulate this condition of interdependence, I define the upper and lower asymptotic elasticities of the production function ($\bar{\alpha}$ and $\underline{\alpha}$, respectively and an "effective" utility function. It is shown that if (a) the utility function is bounded above, (b) $\underline{\alpha} > 1 - \underline{\alpha}$, and (c) there is $0 < \alpha^* < \underline{\alpha}$ for which the * area under the α^*-effective utility function is finite, then an optimal program exists (Theorem 5.1).[6] It is also shown that if an optimal program exists, then the utility function is

[4] It is worth mentioning that in traditional models, when future utilities are *discounted*, optimality is characterized by the competitive conditions, and the transversality condition that the value of inputs converges to zero. See, for example, Peleg and Ryder [**16**] for a statement of this result.

[5] These asymptotic properties were observed by Solow [**18**] and Dasgupta and Heal [**5**], in the case where the production function is Cobb-Douglas, and the utility function has a constant elasticity of marginal utility. Thus, Theorem 4.1 is a generalization of their results to cases where the production and utility functions assume nonparametric forms.

[6] For precise definitions of the concepts used, and an accurate statement of the Theorem, see Section 5.

bounded above (Lemma 6.1). Then, for bounded utility functions, for which the lower asymptotic elasticity is positive (see Assumption 9), it is shown that if either $(a')\bar{\alpha} < 1-\bar{\alpha}$ or (b') for some $\alpha > \bar{\alpha}$, the * area under the α-effective utility

2. THE MODEL

2a. *Production*

Consider an economy, with a technology given by a production function, G, from R_+^2 to R_+. The production possibilities consist of capital input, k, exhaustible resource input, r, and current output $G(k, r)$ for $(k, r) \geqslant 0$.[8] Capital is assumed to be nondepreciating,[9] and total output, y, is defined as $[G(k, r)+k]$ for $(k, r) \geqslant 0$. A total output function, F, can then be defined by

$$(2.1) \qquad F(k, r) = G(k, r) + k \qquad \text{for} \quad (k, r) \geqslant 0.$$

The production function, G, is assumed to satisfy:

ASSUMPTION 1: *G is concave, homogeneous of degree one, continuous for $(k, r) \geqslant 0$, and twice differentiable for $(k, r) \gg 0$.*

ASSUMPTION 2: $G_k = (\partial G/\partial k) > 0$, $G_r = (\partial G/\partial r) > 0$, *for* $(k, r) \gg 0$.

The initial capital input, $\underset{\sim}{k}$, and the initial available stock of the exhaustible resource, $\underset{\sim}{S}$, are historically given, and positive. A *feasible program* is a sequence $\langle k, r, y, c\rangle = \langle k_t, r_t, y_{t+1}, c_{t+1}\rangle$ such that

$$(2.2) \qquad k_0 = \underset{\sim}{k}, \qquad \sum_{t=0}^{\infty} r_t \leqslant \underset{\sim}{S}, \qquad y_{t+1} = F(k_t, r_t) \qquad \text{for} \quad t \geqslant 0,$$

$$c_{t+1} = y_{t+1} - k_{t+1}, \qquad (k_t, r_t, y_{t+1}, c_{t+1}) \geqslant 0 \qquad \text{for } t \geqslant 0.$$

Associated with a feasible program $\langle k, r, y, c\rangle$ is a sequence of *resource stocks* $\langle S\rangle = \langle S_t\rangle$, given by

$$(2.3) \qquad S_0 = \underset{\sim}{S}, \qquad S_{t+1} = S_t - r_t \qquad \text{for} \quad t \geqslant 0.$$

[7] For a precise statement of this result, see Section 6. It is helpful to look, in this connection, at Theorem 6.1, as the corollary follows directly from this result.

[8] For any two n-vectors, a and b, $a \geqslant b$ means $a_i \geqslant b_i$, for $i = 1, \ldots, n$; $a > b$ means $a \geqslant b$, and $a \neq b$; $a \gg b$ means $a_i > b_i$, for $i = 1, \ldots, n$.

[9] It should be emphasized that, as in Solow [**18**], Stiglitz [**19**], Dasgupta and Heal [**5**], this paper relies on the assumption that there is no depreciation of capital. Alternatively, one might start with a "gross-output function" $H(k, r)$ from R_+^2 to R_+, and a growth equation

$$H(k_t, r_t) = (k_{t+1} - k_t) + \delta k_t + c_{t+1}$$

where $0 \leqslant \delta \leqslant 1$. Then, defining the "net-output function" $G(k, r) = H(k, r) - \delta k$, for $(k, r) \geqslant 0$, we get the model described in Section 2. However, while the nonnegative range of H is easily justified, the nonnegative range of G—which is assumed in Section 2—requires either (i) $\delta = 0$, or (ii) if $\delta > 0$, then $H(k, 0) \geqq \delta k$ for $k \geqslant 0$. Condition (ii) requires that without the exhaustible resource input, the gross output can at least cover depreciation expenses. It should also be noted that in case (ii), under Assumption 7, we, in fact, require $H(k, 0) = \delta k$ for $k > 0$.

Notice that, by (2.2), $S_t \geqslant 0$, for $t \geqslant 0$. A feasible program $\langle k, r, y, c\rangle$ is called *interior* if $(k_t, r_t) \gg 0$, for $t \geqslant 0$. It is said to *maintain a positive consumption level* if $\inf_{t\geqslant 1} c_t > 0$.

A feasible program $\langle \bar{k}, \bar{r}, \bar{y}, \bar{c}\rangle$ dominates a feasible program $\langle k, r, y, c\rangle$ if $\bar{c}_t \geqslant c_t$ for all $t \geqslant 1$, and $\bar{c}_t > c_t$ for some t. A feasible program $\langle k, r, y, c\rangle$ is *inefficient* if there is a feasible program $\langle \bar{k}, \bar{r}, \bar{y}, \bar{c}\rangle$ which dominates it. It is *efficient* if it is not inefficient.

2b. *Preferences*

The planner is endowed with a utility function, u, from R_+ to $\tilde{R}$.[10] A feasible program $\langle k^*, r^*, y^*, c^*\rangle$ is called *optimal* if

(2.4) $$\lim_{T\to\infty} \sup \sum_{t=1}^{T} [u(c_t) - u(c_t^*)] \leqslant 0$$

for every feasible program $\langle k, r, y, c\rangle$.

A feasible program $\langle k^*, r^*, y^*, c^*\rangle$ is called *competitive* if there is a sequence $\langle p^*, q^*\rangle = \langle p_t^*, q_t^*\rangle$ of nonnegative prices, such that

(2.5) $$u(c_t^*) - p_t^* c_t^* \geqslant u(c) - p_t^* c, \qquad \text{for} \quad c \geqslant 0,\ t \geqslant 1,$$

(2.6) $$p_{t+1}^* y_{t+1}^* - p_t^* k_t^* - q_t^* r_t^* \geqslant p_{t+1}^* y - p_t^* k - q_t^* r$$
$$\text{for} \qquad (k, r) \geqslant 0,\ y = F(k, r),\ t \geqslant 0,$$

(2.7) $$q_t^* = q_{t+1}^* \qquad \text{for} \quad t \geqslant 0.$$

A price sequence $\langle p^*, q^*\rangle$, associated with a competitive program $\langle k^*, r^*, y^*, c^*\rangle$ for which (2.5), (2.6), (2.7) hold, are called *competitive prices*; (2.5), (2.6), (2.7) are called *competitive conditions*. A competitive program $\langle k^*, r^*, y^*, c^*\rangle$ satisfies the *transversality condition* at the price sequence $\langle p^*, q^*\rangle$ if

(2.8) $$\lim_{t\to\infty} [p_t^* k_t^* + q_t^* S_t^*] = 0.$$

The following assumptions on u will be used in this paper:

ASSUMPTION 3: *$u(c)$ is strictly increasing for $c \geqslant 0$.*

ASSUMPTION 4: *$u(c)$ is twice differentiable for $c > 0$.*

ASSUMPTION 5: *$u(c)$ is concave for $c \geqslant 0$; $u''(c) < 0$ for $c > 0$.*

ASSUMPTION 6: *$u(c)$ satisfies the end-point condition: $u'(c) \to \infty$ as $c \to 0$.*

[10] $\tilde{R}$ denotes the extended real line, that is $[-\infty, \infty]$. This is to allow utility functions for which the utility level goes to minus infinity, as consumption goes to zero.

3. CHARACTERIZATION OF OPTIMALITY

A problem of long-standing interest in the theory of optimal economic growth is to find suitable conditions, which completely characterize the set of optimal programs. This is usually referred to as the problem of characterizing optimality.

In this section, I will show that a feasible program is optimal if and only if (a) it is competitive, and (b) it satisfies the transversality condition at its competitive prices (Theorem 3.1). A consequence of this result is that a competitive program is optimal if and only if it satisfies the transversality condition (see Corollary 3.1).

It should be noted that the traditional theory of optimal economic growth, where exhaustible resources are not present as essential factors of production (and where future utilities are not discounted, as in the present paper), observes that, under suitable conditions on the technology, if the value of inputs along a competitive program is bounded above, then this program is optimal (see, for example, von Weizsacker **[20]** and McKenzie **[11]**, for the case where technology is allowed to change over time). Peleg **[14, 15]** shows that when the technology set is stationary over time, the converse is also true.

The introduction of the exhaustible resource as an essential factor of production into the model, therefore, seems to produce the following qualitative difference in the characterization of optimality: whether or not competitive programs are optimal is signalled by the transversality condition being satisfied or violated, rather than by the input value boundedness condition being satisfied, or violated.[11]

The exhaustible resource will be called an *essential* factor of production if $G(k, 0) = 0$, for $k \geq 0$. I will now assume:

ASSUMPTION 7: *$G(k, 0) = 0 = G(0, r)$, for $(k, r) \geq 0$; $G_r \to \infty$ as $r \to 0$, for $k > 0$.*

THEOREM 3.1: *Under Assumptions 1–7 a feasible program $\langle k^*, r^*, y^*, c^*\rangle$ is optimal if and only if there is a sequence $\langle p^*, q^*\rangle$ of nonnegative prices, satisfying (2.5), (2.6), (2.7) and (2.8).*[12]

PROOF: *Necessity.* Suppose a feasible program $\langle k^*, r^*, y^*, c^*\rangle$ is optimal. Then, for each $t \geq 1$, the expression $u[F(k^*_{t-1}, r^*_{t-1}) - k] + u[F(k, r^*_t) - k^*_{t+1}]$ must be a maximum at $k = k^*_t$. By Assumption 6, the maximum must be at an interior point; that is, $c^*_t > 0$ for $t \geq 1$. Hence, by Assumption 7, $k^*_t > 0$ and $r^*_t > 0$ for $t \geq 0$. So, using Assumptions 1 and 4, we get

$$(3.1) \qquad u'(c^*_t) = u'(c^*_{t+1})F_{k^*_t} \qquad \text{for} \quad t \geq 1.$$

[11] This is closely related to the qualitative difference in the characterization of *efficiency* that is produced by including or excluding an exhaustible resource as an essential factor of production. For a detailed discussion of this point, see Mitra **[12]**.

[12] Notice that the present value price of the exhaustible resource is constant for an optimal program. In fact, this is true of any *efficient* program. A discussion of this, and related results, are omitted here, as they appear, in detail, in Mitra **[12]**.

Also, for each $t \geq 1$, for $0 < r \leq r^*_{t-1} + r^*_t$, the expression

$$u[F(k^*_{t-1}, r) - k^*_t] + u[F(k^*_t, r^*_{t-1} + r^*_t - r) - k^*_{t+1}]$$

must be a maximum at $r = r^*_{t-1}$. Since $r^*_t > 0$, $k^*_t > 0$ for $t \geq 0$, so we get

$$\text{(3.2)} \qquad u'(c^*_t)F_{r^*_{t-1}} = u'(c^*_{t+1})F_{r^*_t} \qquad \text{for} \quad t \geq 1.$$

From (3.1), (3.2), we obtain, using Assumptions 2 and 3,

$$\text{(3.3)} \qquad (F_{r^*_t}/F_{r^*_{t-1}}) = F_{k^*_t} \qquad \text{for} \quad t \geq 1.$$

Define a sequence $\langle p^*, q^* \rangle$ in the following way:

$$\text{(3.4)} \qquad \begin{aligned} &p^*_0 = u'(c^*_1)F_{k^*_0}; \qquad p^*_t = u'(c^*_t) \qquad \text{for} \quad t \geq 1; \\ &q^*_t = u'(c^*_1)F_{r^*_0} \qquad \text{for} \quad t \geq 0. \end{aligned}$$

Now, by Assumption 5, we have for $c \geq 0$ and $t \geq 1$,

$$\text{(3.5)} \qquad u(c) - u(c^*_t) \leq u'(c^*_t)(c - c^*_t) = p^*_t(c - c^*_t).$$

Rearranging the terms in (3.5) yields (2.5). Similarly, by Assumption 1, we have for $(k, r) \geq 0$, $y = F(k, r)$ and $t \geq 0$,

$$\text{(3.6)} \qquad F(k, r) - F(k^*_t, r^*_t) \leq F_{k^*_t}(k - k^*_t) + F_{r^*_t}(r - r^*_t).$$

Multiplying through by p^*_{t+1} in (3.6), and using (3.1), (3.2), we get

$$\text{(3.7)} \qquad p^*_{t+1}(y - y^*_{t+1}) \leq p^*_t(k - k^*_t) + q^*_t(r - r^*_t).$$

Rearranging the terms in (3.7) yields (2.6). Finally, by (3.4), $q^*_t = q^*_{t+1}$ for $t \geq 0$, which yields (2.7). Thus, we have found a sequence $\langle p^*, q^* \rangle$ of nonnegative prices, such that (2.5), (2.6), and (2.7) are satisfied. It remains to show that at these prices, the transversality condition (2.8) is also satisfied.[13]

Note that since $\langle k^*, r^*, y^*, c^* \rangle$ is optimal, so by Assumption 3, it is efficient. By (2.3), S^*_t is monotonically nonincreasing, and bounded below by zero, so S^*_t converges to a limit; call it $\bar{S}$. Then $\bar{S} \geq 0$. We claim that $\bar{S} = 0$. For, if $\bar{S} > 0$, then we could use an extra $\bar{S}$ amount of the resource in period $t = 0$, produce and consume more in period $t = 1$ (by Assumption 2), leaving the rest of the program unaffected. This would violate efficiency. Hence $\bar{S} = 0$ or $q^*_t S^*_t (= q^*_0 S^*_t)$ converges to zero as $t \to \infty$. So, to establish (2.8), we have only to establish that $p^*_t k^*_t \to 0$ as $t \to \infty$.

First, we prove that $p^*_t k^*_t$ converges to a limit. For $t \geq 0$,

$$\begin{aligned} p^*_{t+1}c^*_{t+1} &= p^*_{t+1}y^*_{t+1} - p^*_{t+1}k^*_{t+1} \\ &= [p^*_{t+1}y^*_{t+1} - p^*_t k^*_t - q^*_t r^*_t] \\ &\quad + [p^*_t k^*_t - p^*_{t+1}k^*_{t+1}] + q^*_t r^*_t \\ &= [p^*_t k^*_t - p^*_{t+1}k^*_{t+1}] + q^*_t r^*_t \end{aligned}$$

[13] This proof follows closely the method used in Mitra [**12**].

by using (2.6), and the fact that $G(k, r)$ and hence $F(k, r)$ is homogeneous of degree one. Hence, for $T \geq 0$,

$$(3.8) \qquad \sum_{t=0}^{T} p_{t+1}^* c_{t+1}^* = \sum_{t=0}^{T} [p_t^* k_t^* - p_{t+1}^* k_{t+1}^*] + \sum_{t=0}^{T} q_t^* r_t^*$$
$$= p_0^* k_0^* + \sum_{t=0}^{T} q_t^* r_t^* - p_{T+1}^* k_{T+1}^*.$$

Now, $\Sigma_{t=0}^{T} p_{t+1}^* c_{t+1}^*$ is monotonically nondecreasing in T. Also, by (2.7),

$$(3.9) \qquad \sum_{t=0}^{T} q_t^* r_t^* = q_0^* \sum_{t=0}^{T} r_t^* \leq q_0^* \underset{\sim}{S}.$$

So, by (3.8), $\Sigma_{t=0}^{T} p_{t+1}^* c_{t+1}^*$ is bounded above by $p_0^* k + q_0^* \underset{\sim}{S}$ for $T \geq 0$. Hence, $\Sigma_{t=0}^{\infty} p_{t+1}^* c_{t+1}^*$ converges. By (3.9), $\Sigma_{t=0}^{\infty} q_t^* r_t^*$ also converges. Hence, by (3.8), $p_{T+1}^* k_{T+1}^*$ converges to a limit, as $T \to \infty$.

We claim, now, that this limit is zero. If not, then there is $b > 0$, such that for $t \geq 0$, $p_t^* k_t^* \geq b$. Since $\Sigma_{t=0}^{\infty} p_{t+1}^* c_{t+1}^*$ converges, there is τ such that $C_\tau^* \equiv \Sigma_{t=\tau}^{\infty} p_{t+1}^* c_{t+1}^* \leq (b/2)$. If $C_\tau^* = 0$, the program $\langle k^*, r^*, y^*, c^* \rangle$ is clearly inefficient. So we consider only the case in which $C_\tau^* > 0$. Write $\nu_{t+1}^* \equiv [(p_{t+1}^* c_{t+1}^*)/C_\tau^*]$ for $t \geq \tau$. Clearly, $\Sigma_{t=\tau}^{\infty} \nu_{t+1}^* = 1$. Now, construct a sequence $\langle k, r, y, c \rangle$ in the following way: $(k_0, r_0) = (k_0^*, r_0^*)$; for $1 \leq t < \tau$, $(k_t, y_t, r_t, c_t) = (k_t^*, r_t^*, y_t^*, c_t^*)$; for $t = \tau$, $(k_t, r_t) = \frac{1}{2}(k_t^*, r_t^*)$, $c_t = c_t^* + \frac{1}{2}k_t^*$, $y_t = y_t^*$; for $t > \tau$, $(k_t, r_t) = a_t^*(k_t^*, r_t^*)$, $y_t = F(k_{t-1}, r_{t-1})$, $c_t = y_t - k_t$, where $a_t^* = [\frac{1}{2} - \frac{1}{2}\Sigma_{s=\tau}^{t-1} \nu_{s+1}^*]$.

Now, clearly, $(k_t, r_t) \gg 0$ for $t \geq 0$; $c_t = c_t^*$ for $t < \tau$, and $c_t > c_t^*$ for $t = \tau$. We will show that $c_t \geq 0$ for $t \geq \tau$ (so that the sequence $\langle k, r, y, c \rangle$ represents a feasible program), and that $c_t \geq c_t^*$ for $t > \tau$ (so that $\langle k, r, y, c \rangle$ dominates $\langle k^*, r^*, y^*, c^* \rangle$).

For $t > \tau$,

$$\begin{aligned} p_t^* c_t &= p_t^* F(k_{t-1}, r_{t-1}) - p_t^* k_t \\ &\geq p_t^* F(a_{t-1}^* k_{t-1}^*, a_{t-1}^* r_{t-1}^*) - p_t^* a_t^* k_t^* \\ &= a_{t-1}^* p_t^* F(k_{t-1}^*, r_{t-1}^*) - a_t^* p_t^* k_t^* \\ &= a_{t-1}^* p_t^* (c_t^* + k_t^*) - a_t^* p_t^* k_t^* \\ &= a_{t-1}^* p_t^* c_t^* + (1 - a_{t-1}^*) p_t^* c_t^* + a_{t-1}^* p_t^* k_t^* - a_t^* p_t^* k_t^* \\ &\qquad - (1 - a_{t-1}^*) p_t^* c_t^* \\ &\geq p_t^* c_t^* + p_t^* k_t^* [a_{t-1}^* - a_t^* - \tfrac{1}{2}(1 - a_{t-1}^*) \nu_t^*] \\ &\geq p_t^* c_t^* + p_t^* k_t^* [\tfrac{1}{2}\nu_t^* - \tfrac{1}{2}\nu_t^*] \\ &= p_t^* c_t^*. \end{aligned}$$

Since $p_t^* > 0$ for $t \geq 0$, we have $c_t \geq c_t^*$ for $t > \tau$. This proves that $\langle k, r, y, c \rangle$ is a feasible program, and that it dominates $\langle k^*, r^*, y^*, c^* \rangle$, so that the latter is inefficient. This contradiction establishes that $p_t^* k_t^* \to 0$, as $t \to \infty$, and hence (2.8).

PROOF: *Sufficiency*. Suppose a feasible program $\langle k^*, r^*, y^*, c^*\rangle$ satisfies (2.5), (2.6), (2.7), and (2.8). Then given any feasible program $\langle k, r, y, c\rangle$, we have for $t \geq 1$ (by using (2.5), (2.6))

$$\begin{aligned}
u(c_t) - u(c_t^*) &\leq p_t^*(c_t - c_t^*) \\
&= p_t^*(y_t - k_t) - p_t^*(y_t^* - k_t^*) \\
&= [p_t^* y_t - p_{t-1}^* k_{t-1} - q_{t-1}^* r_{t-1}] + q_{t-1}^* r_{t-1} \\
&\quad + [p_{t-1}^* k_{t-1} - p_t^* k_t^*] - [p_t^* y_t^* - p_{t-1}^* k_{t-1}^* - q_{t-1}^* r_{t-1}^*] \\
&\quad - q_{t-1}^* r_{t-1}^* - [p_{t-1}^* k_{t-1}^* - p_t^* k_t^*] \\
&\leq q_{t-1}^*(r_{t-1} - r_{t-1}^*) + [p_{t-1}^* k_{t-1} - p_t^* k_t] - [p_{t-1}^* k_{t-1}^* - p_t^* k_t^*].
\end{aligned}$$

Hence, for $T \geq 1$, we have

$$\sum_{t=1}^{T} [u(c_t) - u(c_t^*)] \leq \sum_{t=1}^{T} q_{t-1}^*(r_{t-1} - r_{t-1}^*) + p_0^*(k_0 - k_0^*) + p_T^*(k_T^* - k_T). \tag{3.10}$$

Using (2.7) in (3.10), we then have

$$\begin{aligned}
\sum_{t=1}^{T} [u(c_t) - u(c_t^*)] &\leq q_0^* \sum_{t=1}^{T} (r_{t-1} - r_{t-1}^*) + p_T^*(k_T^* - k_T) \\
&= q_0^*(\underset{\sim}{S} - S_T) - q_0^*(\underset{\sim}{S} - S_T^*) + p_T^*(k_T^* - k_T) \\
&= [q_0^* S_T^* + p_T^* k_T^*] - [q_0^* S_T + p_T^* k_T^*] \\
&\leq [q_0^* S_T^* + p_T^* S_T^*] = [q_T^* S_T^* + p_T^* k_T^*].
\end{aligned}$$

Using (2.8) in the above inequality, we have

$$\limsup_{T\to\infty} \sum_{t=1}^{T} [u(c_t) - u(c_t^*)] \leq 0. \tag{3.11}$$

This proves that $\langle k^*, r^*, y^*, c^*\rangle$ is optimal.

The following two corollaries are stated, without proofs, since they follow directly from the Proof of Theorem 3.1.

COROLLARY 3.1: *Under Assumptions* 1–7, *if* $\langle k^*, r^*, y^*, c^*\rangle$ *is a competitive program, then the following five statements are equivalent:* (i) $\langle k^*, r^*, y^*, c^*\rangle$ *is optimal.* (ii) $\langle k^*, r^*, y^*, c^*\rangle$ *is efficient.* (iii) $\lim_{t\to\infty}[p_t^* k_t^* + q_t^* S_t^*] = 0$. (iv) $\infty > \sum_{t=1}^{\infty} p_t^* c_t^* = p_0^* \underset{\sim}{k} + q_0^* \underset{\sim}{S}$. (v) $\infty > \sum_{t=1}^{\infty} p_t^* c_t^* \geq \sum_{t=1}^{\infty} p_t^* c_t$ *for every feasible program* $\langle k, r, y, c\rangle$.

COROLLARY 3.2: *Under Assumptions* 1–7, *a feasible program* $\langle k^*, r^*, y^*, c^*\rangle$ *is optimal if and only if*: (i) $(k_t^*, r_t^*, c_{t+1}^*) \gg 0$ *for* $t \geq 0$. (ii) $u'(c_t^*) = u'(c_{t+1}^*) F_{k_t^*}$ *for* $t \geq 1$. (iii) $(F_{r_t^*}/F_{r_{t-1}^*}) = F_{k_t^*}$ *for* $t \geq 1$. (iv) $\lim_{t\to\infty} S_t^* = 0$; $\lim_{t\to\infty}[u'(c_t^*) k_t^*] = 0$.

In Corollary 3.1, statement (iv) says that the present value of the consumption sequence of the competitive program equals the present value of initial "wealth." Statement (v) says that the present value of the consumption sequence of the competitive program is a maximum in the set of all feasible consumption sequences.

In Corollary 3.2, (ii) is the "Ramsey-Euler equation," so familiar in traditional optimal growth theory. Condition (iii) is an efficiency condition for allocation of the exhaustible resource over time, which says that the proportionate rate of change of the marginal productivity of the resource always equals the level of the marginal productivity of capital. The first part of (iv) says that the total resource stock should be exhausted over the horizon; the second part is a "capital-value transversality condition."

In the rest of the paper, in view of Theorem 3.1, we will always associate with an optimal program a price sequence given precisely by (3.4).

4. ASYMPTOTIC PROPERTIES OF OPTIMAL PROGRAMS

In this section, I will establish some asymptotic properties of optimal depletion programs. Two of the important properties are: (a) the consumption level along the optimal program monotonically increases to infinity; (b) the relative price of the consumption good to that of the exhaustible resource monotonically decreases to zero. The first property shows that, for optimal programs (when they exist) capital accumulation will more than offset the effects of a (rapidly) deteriorating resource stock, and make larger future consumption levels possible. The second property simply states that as the resource stock gets depleted over time the price of the resource relative to that of the consumption good must rise to infinitely high levels, reflecting the increased valuation placed on this essential but exhaustible resource. Similar results have been obtained by Solow [**18**], and Dasgupta and Heal [**5**], in the case where the production function is of the Cobb-Douglas form, and the utility function satisfies the condition that the elasticity of marginal utility is constant.

It should be noted that asymptotic properties of optimal programs, when future utilities are discounted (see, for example, Dasgupta [**3**], Dasgupta and Heal [**4**, **5**], and Stiglitz [**19**]), are significantly different from those in the undiscounted case, the most important being that in the discounted case the level of consumption along the optimal program can decrease to zero.

I will start by defining a new term. For $k > 0$, the *resource-capital ratio* is defined by

(4.1) $$z = (r/k).$$

It is of course clear that, by Assumption 1, $G(k, r) = k\,G(1, z) = k\,g(z)$. Furthermore, we have $G_k = g(z) - z\,g'(z)$, and $G_r = g'(z)$, so that the marginal products of capital and resource depend only on the resource-capital ratio.

THEOREM 4.1:[14] *Under Assumptions 1–7 if* $\langle k^*, r^*, y^*, c^*\rangle$ *is an optimal program, then*: (i) $c_t^*, k_t^*, y_t^*, G_{r_t^*}$ *monotonically increase to* ∞, *as* $t\to\infty$; (ii) $S_t^*, z_t^*, (p_t^*/q_t^*)$, $G_{k_t^*}$ *monotonically decrease to* 0, *as* $t\to\infty$; (iii) $G(k_t^*, r_t^*)\to\infty$, *as* $t\to\infty$; (iv) r_t^*, $[G(k_t^*, r_t^*)/k_t^*]$, $[c_{t+1}^*/k_t^*]$ *converge to zero, as* $t\to\infty$.

PROOF: To prove the theorem, we will use Corollary 3.2. We start with statement (i). We know that for $t\geq 1$, $u'(c_t^*)=u'(c_{t+1}^*)(1+G_{k_t^*})$, so that $u'(c_t^*)>u'(c_{t+1}^*)$, and $c_{t+1}^*>c_t^*$ for $t\geq 1$. We claim that $c_t^*\to\infty$, as $t\to\infty$. If not, then $c_t^*\to\bar{c}$, where $0<\bar{c}<\infty$. This implies, by a direct application of condition (iv) of Corollary 3.2, that $k_t^*\to 0$ as $t\to\infty$. This, in turn, implies that $c_{t+1}^*\to 0$ as $t\to\infty$, by feasibility and Assumption 7, noting that $r_t^*\leq \underset{\sim}{S}$ for $t\geq 0$. This contradiction establishes the claim. Hence c_t^* monotonically increases to ∞, as $t\to\infty$.

It follows from condition (iii) of Corollary 3.2, that $g'(z_t^*)=G_{r_t^*}>G_{r_{t-1}^*}=g'(z_{t-1}^*)$ for $t\geq 1$, so that $z_t^*\leq z_{t-1}^*$ for $t\geq 1$. Suppose for some τ, $k_\tau^*\geq k_{\tau+1}^*$. Then, since $z_\tau^*\leq z_\tau^*$, so $r_{\tau+1}^*\leq r_\tau^*$. Also, since $c_{\tau+1}^*<c_{\tau+2}^*$, so $k_{\tau+1}^*\geq k_{\tau+2}^*$. Repeating the argument, we observe that k_t^* is bounded above by k_τ^*, for $t\geq\tau$. But this violates feasibility, since $r_t^*\leq \underset{\sim}{S}$ for $t\geq 0$, and $c_t^*\to\infty$ as $t\to\infty$. Hence, k_t^* must be monotonically increasing. Furthermore, $k_t^*\to\infty$, as $t\to\infty$, for if $k_t^*\leq\bar{k}$, for a subsequence of periods, then c_{t+1}^* must be bounded for that subsequence too, by feasibility, and the fact that $r_t^*\leq \underset{\sim}{S}$, for $t\geq 0$. This contradiction establishes the result regarding k_t^*.

Since c_t^* monotonically increases to ∞, and k_t^* monotonically increases to ∞, as $t\to\infty$, so $y_t^*=k_t^*+c_t^*$ monotonically increases to ∞ as $t\to\infty$. Finally, note that by condition (iii) of Corollary 3.2, that $G_{r_t^*}$ is monotonically increasing. Also, since $r_t^*\leq \underset{\sim}{S}$ for $t\geq 0$, and k_t^* increases to infinity, so $z_t^*\to 0$, as $t\to\infty$. So, $G_{r_t^*}=g'(z_t^*)\to\infty$ as $t\to\infty$ by Assumption 7. This completes the proof of statement (i).

We come, now, to the proof of statement (ii). Notice that since $r_t^*>0$ for $t\geq 0$, so $S_{t+1}^*<S_t^*$ for $t\geq 0$. Also, by condition (iv) of Corollary 3.2, $S_t^*\to 0$ as $t\to\infty$. Hence S_t^* monotonically decreases to zero. We have already noted that by condition (iii) of Corollary 3.2, $g'(z_{t+1}^*)>g'(z_t^*)$, so that $z_{t+1}^*<z_t^*$ for $t\geq 0$. Also, $r_t^*\leq \underset{\sim}{S}$ for $t\geq 0$, while $k_t^*\to\infty$ as $t\to\infty$, so that z_t^* monotonically decreases to zero. Since c_t^* monotonically increases to infinity, p_t^* monotonically decreases to zero, by Assumptions 5 and 6. Also, q_t^* is a constant, so (p_t^*/q_t^*) monotonically decreases to zero. Finally, note that $G_{k_t^*}=g(z_t^*)-z_t^*g'(z_t^*)$, which decreases, since z_t^* decreases as $t\to\infty$. Also, by Assumption 7, $g(z_t^*)\to 0$ as $z_t^*\to 0$, so that $G_{k_t^*}$ monotonically decreases to zero at $t\to\infty$. This completes the proof of statement (ii).

To prove statement (iii), notice that $G(k_t^*, r_t^*)=c_{t+1}^*+(k_{t+1}^*-k_t^*)\geq c_{t+1}^*$, since k_t^* is monotonically increasing. Hence, noting that c_t^* increases to infinity as $t\to\infty$, $G(k_t^*, r_t^*)$ must also become infinitely large as $t\to\infty$. This completes the proof of statement (iii).

[14] It should be noted that Solow [**18**] and Dasgupta and Heal [**5**] are able to characterize the asymptotic behavior of the following variables as well: the ratio of consumption to current output, and the ratio of the resource flow to the remaining resource stock. I believe this is possible because they use particular parametric forms for production and utility functions, while Theorem 4.1 deals with general forms of these functions.

We finally come to the proof of statement (iv). Notice that since $\Sigma_{t=0}^{\infty} r_t^* \leqslant \underset{\sim}{S}$, so $r_t^* \to 0$ as $t \to \infty$. Next, note that $g(z_t^*) = [G(k_t^*, r_t^*)/k_t^*]$, and we know that $z_t^* \to 0$ as $t \to \infty$. So, by Assumption 7, $[G(k_t^*, r_t^*)/k_t^*] \to 0$ as $t \to \infty$. Finally, note that $c_{t+1}^* = G(k_t^*, r_t^*) + (k_t^* - k_{t+1}^*) \leqslant G(k_t^*, r_t^*)$, since k_t^* is monotonically increasing as $t \to \infty$. Hence $[c_{t+1}^*/k_t^*] \leqslant [G(k_t^*, r_t^*)/k_t^*]$, so that, by the immediately previous result, $[c_{t+1}^*/k_t^*] \to 0$ as $t \to \infty$. This completes the proof of statement (iv).

5. AN EXISTENCE RESULT

This section is devoted to finding a set of conditions on the utility and production functions, which ensure that there exists an optimal program. It should be noted that both Solow [**18**] and Dasgupta and Heal [**5**], prove the existence of an optimal program (when future utilities are not discounted), by "constructing" a program, and showing that such a program is indeed optimal. Their method of construction is possible because of the parametric forms that they use for the utility and production functions. In the more general framework of this paper, alternative methods of proof have to be devised.

Consider a "traditional" method of proving the existence of optimal programs in models which do not emphasize the role of an exhaustible resource as an essential factor of production (see, for example, Koopmans [**9**], Gale [**6**], Brock [**1**], McKenzie [**10**]). There are primarily three steps in the argument: (a) there exists a "good" program; (b) any feasible program which is not "good" is ineligible as a candidate for an optimal program (and, therefore, "bad"); (c) in the class of good programs, one finds a program which maximizes the sum of utilities (or shortfalls from some level), and this is an optimal program.

Now, it appears that in these traditional models, because all stocks of goods can be ultimately augmented, and because of the useful properties of a golden-rule program (or, its multisectoral counterpart, the optimal stationary program), part (a) is relatively easy to verify. As soon as exhaustible resources are introduced into the model, the stock of the exhaustible resource clearly cannot be augmented, and the golden-rule program ceases to exist. Consequently, it turns out that in proving the existence of optimal programs, it is part (a) of the three-step argument which is of crucial importance. One must ensure that even with exhaustible resource constraints, consumption can be increased to infinity fast enough to make the utility sum converge. If this can be done, part (b) is trivial to verify, and part (c) follows by traditional arguments. Thus, the key to an existence result is to impose sufficient conditions on the utility and production functions to ensure that a good program exists.

I will start by defining a few terms. For $r > 0$, the *capital-resource ratio* is defined as

$$x = (k/r). \tag{5.1}$$

It is, of course, clear that $G(k, r) = rG(x, 1) = rf(x)$, and $G_k = f'(x)$, $G_r = f(x) - x f'(x)$. The *share of capital in current output* or the *elasticity* of f is defined by

$$e(x) = [f'(x)\, x/f(x)] \quad \text{for} \quad x > 0. \tag{5.2}$$

Following Brock and Gale [**2**], I define the *asymptotic upper and lower elasticities of* f ($\bar{\alpha}$ and $\underline{\alpha}$, respectively) by

$$(5.3) \qquad \bar{\alpha} = \limsup_{x\to\infty} e(x); \qquad \underline{\alpha} = \liminf_{x\to\infty} e(x).$$

Given any α such that $0<\alpha<1$, I will denote $(1-\alpha)$ by β; $(1-\beta)/(1-\alpha)$ by γ; $(\alpha-\beta)/(1-\alpha)$ by δ; $(\alpha-\beta)/(1-\beta)$ by ε.

The utility function, u, is said to be *bounded above if*

$$(5.4) \qquad \sup_{c\geqslant 0} u(c)<\infty.$$

I will, now, proceed to assume the following:

ASSUMPTION 8: *u is bounded above.*

Under Assumption 8 let us denote $\sup_{c\geqslant 0} u(c)$ by $\bar{u}$. Without loss of generality, we can normalize $\bar{u}=0$. Regarding the justification of Assumption 8, it should be pointed out that a *necessary condition* for the existence of an optimal program is that u be bounded above (see Lemma 6.1 in the next section). Thus, without Assumption 8 we clearly cannot have an existence theorem.

Given Assumption 8, I define for $0<\alpha<1$, *an α-effective utility function* $w(v;\alpha)$, for $v\geqslant 0$ by

$$(5.5) \qquad w(v;\alpha) = -u(v^{\delta}).$$

The * *area under the α-effective utility functions* is defined by

$$(5.6) \qquad \int_1^{\infty} w(v;\alpha)\,dv.$$

A feasible program $\langle k, r, y, c\rangle$ is called *good* if there is $M>-\infty$, such that

$$(5.7) \qquad \sum_{t=1}^{T} u(c_t) \geqslant M \qquad \text{for} \quad T\geqslant 1.$$

It is called *bad* if

$$(5.8) \qquad \sum_{t=1}^{T} u(c_t) \to -\infty \qquad \text{as} \quad T\to\infty.$$

LEMMA 5.1: *Under Assumptions* 1–7 *and* $\underline{\alpha}>\beta$, *given any* α^* *satisfying* $0<\alpha^*<\underline{\alpha}$, *there exists a feasible program* $\langle k, r, y, c\rangle$ *and an integer* $T\geqslant 1$ *such that*

$$(5.9) \qquad c_t>0 \qquad \textit{for} \quad t\geqslant 1,$$
$$c_t \geqslant t^{\delta^*} \qquad \textit{for} \quad t\geqslant T.$$

PROOF: Choose α such that $\alpha^*<\alpha<\underline{\alpha}$, where α is sufficiently close to $\underline{\alpha}$ to ensure $\alpha>\beta$. Since $\underline{\alpha}>\beta$, this is possible. Notice that since $\alpha>\alpha^*$, so $\delta>\delta^*$.

Choose $1<\mu<\gamma$, where μ is close enough to γ to ensure that $(\mu-1)>\delta^*$. Since $(\gamma-1)=\delta>\delta^*$, this can be done. Finally, choose $\theta>1$, where θ is sufficiently close to 1, such that $(1-\theta\beta)>\mu(1-\alpha)$. Since $(1-\beta)=\gamma(1-\alpha)>\mu(1-\alpha)$, this can be done. Hence, we have

$$(5.10)\qquad \lambda\equiv(\alpha\mu-\theta\beta)>(\mu-1)>\delta^*.$$

By the definition of $\underset{\sim}{\alpha}$ there is $\bar{x}<\infty$ such that, for $x\geqslant\bar{x}$, $f(x)>x^{\alpha}$ (see Brock and Gale [**2**, pp. 234–235], for details). Now, choose $n>0$ such that

$$(5.11)\qquad \sum_{t=0}^{\infty}[n/(1+t)^{\theta}]\leqslant\underset{\sim}{S}.$$

Define a sequence $\langle k', r', y', c'\rangle$ as follows: $k'_t=\underset{\sim}{k}$, $r'_t=[n/(t+1)^{\theta}]$, $y'_{t+1}=G(k'_t, r'_t)+k'_t$, $c'_{t+1}=G(k'_t, r'_t)$ for $t\geqslant 0$. It is clear that $\langle k', r', y', c'\rangle$ is a feasible program, and $x'_t\to\infty$ as $t\to\infty$. Choose $\tau<\infty$ such that the following conditions are satisfied:

(a) $\qquad x'_t>\bar{x} \qquad$ for $\quad t\geqslant\tau$,

(b) $\qquad \frac{1}{2}\underset{\sim}{k}^{\alpha-1}n^{\beta}\geqslant[\mu(t+2)^{\mu-1}]/(t+1)^{\lambda} \quad$ for $\quad t\geqslant\tau$.

It is easy to ensure that (a) is satisfied. Using (5.10), (b) can be ensured as well. Choose $N>0$ such that $N(\tau+1)^{\mu}=\underset{\sim}{k}$; also, let $h=\frac{1}{2}N^{\alpha}n^{\beta}$. Now, define a sequence $\langle k, r, y, c\rangle$ as follows: $k_t=k'_t$ for $t<\tau$, $k_t=N(t+1)^{\mu}$ for $t\geqslant\tau$; $r_t=r'_t$, $y_{t+1}=G(k_t, r_t)+k_t$ for $t\geqslant 0$; $c_{t+1}=c'_{t+1}$ for $t<\tau$; $c_{t+1}=y_{t+1}-k_{t+1}$ for $t\geqslant\tau$. We have to check that $c_{t+1}\geqslant 0$ for $t\geqslant\tau$, to ensure that $\langle k, r, y, c\rangle$ is a feasible program. In fact, we will show that $c_{t+1}\geqslant h(t+1)^{\lambda}$ for $t\geqslant\tau$.

For $t\geqslant\tau$, we have $c_{t+1}=G(k_t, r_t)+k_t-k_{t+1}=r_t f(x_t)+k_t-k_{t+1}>k_t^{\alpha}r_t^{\beta}+k_t-k_{t+1}$ (by choice of $\bar{x}$ and τ). Hence, we have

$$\begin{aligned}
c_{t+1}&=n^{\beta}N^{\alpha}(t+1)^{\lambda}+N(t+1)^{\mu}-N(t+2)^{\mu}\\
&\geqslant N(t+1)^{\lambda}[n^{\beta}N^{\alpha-1}-\mu\{(t+2)^{\mu-1}/(t+1)^{\lambda}\}]\\
&\geqslant N(t+1)^{\lambda}[\tfrac{1}{2}n^{\beta}N^{\alpha-1}+\tfrac{1}{2}n^{\beta}\underset{\sim}{k}^{\alpha-1}-\{\mu(t+2)^{\mu-1}/(t+1)^{\lambda}\}]\\
&\geqslant N(t+1)^{\lambda}\tfrac{1}{2}n^{\beta}N^{\alpha-1}\quad\text{(by choice of }\tau)\\
&=\tfrac{1}{2}N^{\alpha}n^{\beta}(t+1)^{\lambda}=h(t+1)^{\lambda}.
\end{aligned}$$

Hence $\langle k, r, y, c\rangle$ is a feasible program. Since $\lambda>\delta^*$ by (5.10), so there exists an integer $T>\tau$, such that

$$c_t\geqslant t^{\delta^*}\qquad\text{for}\quad t\geqslant T.$$

This establishes the lemma. *Q.E.D.*

LEMMA 5.2: *Under Assumptions* 1–8 *there exists a good program* $\langle k, r, y, c\rangle$ *if*

$$(5.12)\qquad \underline{\alpha} > \underline{\beta}^{15} \quad \textit{and}$$

$$(5.13)\qquad \int_1^{\infty} w(v;\alpha^*)dv < \infty$$

for some α^*, *satisfying* $0 < \alpha^* < \underline{\alpha}$.

PROOF: Given (5.12) and a choice of α^* satisfying (5.13), there is a feasible program $\langle k, r, y, c\rangle$, and an integer $T \geq 1$, such that (5.9) holds (by Lemma 5.1).

Using (5.13), we have, by the Maclaurin–Cauchy integral test (see Hardy [**7**, pp. 351–352]),

$$(5.14)\qquad \sum_{t=1}^{\infty} w[t;\alpha^*] < \infty.$$

This means, by definition of w, that

$$(5.15)\qquad \sum_{t=1}^{\infty} -u[t^{\delta^*}] < \infty.$$

Since $c_t \geq t^{\delta^*}$ for $t \geq T$, so

$$(5.16)\qquad \sum_{t=T}^{\infty} -u(c_t) \leq \sum_{t=T}^{\infty} -u(t^{\delta^*}) \leq \sum_{t=1}^{\infty} -u(t^{\delta^*}) < \infty.$$

Since $c_t > 0$ for $t \geq 1$, so

$$(5.17)\qquad \sum_{t=1}^{\infty} -u(c_t) < \infty.$$

This means that $\langle k, r, y, c\rangle$ is a good program.

LEMMA 5.3: *Under Assumptions* 1–8 *a feasible program* $\langle k, r, y, c\rangle$ *which is not good, is bad.*

PROOF: Suppose $\langle k, r, y, c\rangle$ is a feasible program, which is not good. Then, there is a subsequence of periods T_s for which

$$(5.18)\qquad \sum_{t=1}^{T_s} u(c_t) \to -\infty \quad \text{as} \quad T_s \to \infty.$$

[15] It was shown by Solow [**18**] that if the elasticity of f is constant, then this condition is necessary and sufficient for a positive consumption level to be maintained. My own investigation of this question [**13**], when the production function is of a general form, shows that this condition is *sufficient* to ensure that a positive consumption level be maintained. These results partly justify using this condition to make the problem of more interest. We will also see that a condition "close to" (5.12) is *necessary* for the existence of optimal programs (see (6.6) below).

Since, for $c \geqslant 0$, $u(c) \leqslant 0$, so $\sum_{t=1}^{T} u(c_t)$ is monotonically nonincreasing, and so by (5.18),

(5.19) $$\sum_{t=1}^{T} u(c_t) \to -\infty \quad \text{as} \quad T \to \infty.$$

(5.19) means that $\langle k, r, y, c \rangle$ is a bad program.

THEOREM 5.1: *Under Assumptions* 1–8 *there exists an optimal program if* (5.12) *and* (5.13) *hold.*

PROOF: Define $A = \inf[\Sigma_{t=1}^{\infty} -u(c_t)$: $\langle k, r, y, c \rangle$ is a feasible program]. By Lemma 5.2, there exists a good program, so that $A < \infty$.

Choose a sequence of programs $\langle k^i, r^i, y^i, c^i \rangle$ such that

(5.20) $$\sum_{t=1}^{\infty} -u(c_t^i) \leqslant A + (1/i) \qquad (i = 1, 2, \ldots).$$

Let $\underline{k} = \max[G(\underset{\sim}{k}, \underset{\sim}{S}), \underset{\sim}{k}]$. Then, by feasibility, $k_t^i, c_{t+1}^i, y_{t+1}^i \leqslant 2^t \underline{k}$ for $t \geqslant 0$, for all i; also $S_t^i \leqslant \underset{\sim}{S}$ for $t \geqslant 0$, for all i. Hence, there is a subsequence (call it j) of i, such that for each $t \geqslant 0$, $(k_t^j, S_t^j, y_{t+1}^j, c_{t+1}^j) \to (k_t^*, S_t^*, y_{t+1}^*, c_{t+1}^*)$ as $j \to \infty$. It is straightforward to check that $\langle k^*, r^*, y^*, c^* \rangle$ is a feasible program, by defining $r_t^* = S_t^* - S_{t+1}^*$ for $t \geqslant 0$. Notice that, by the definition of A, $\Sigma_{t=1}^{\infty} -u(c_t^*) \geqslant A$. Let $B = \Sigma_{t=1}^{\infty} -u(c_t^*)$, and let us suppose $B > A$. Pick A_1, B_1 so that $A < A_1 < B_1 < B$. Choose T_0 so that $T \geqslant T_0$ implies $\Sigma_{t=1}^{T} -u(c_t^*) \geqslant B_1$. Notice that for $t \geqslant 1$, $u(c_t^j) \to u(c_t^*)$ as $j \to \infty$. So we can choose j_0, so that $j \geqslant j_0$ implies $\Sigma_{t=1}^{T_0} - u(c_t^j) \geqslant A_1$. This means that we have, for $j \geqslant j_0$,

(5.21) $$A + (1/j) \geqslant \sum_{t=1}^{\infty} -u(c_t^j) \geqslant \sum_{t=1}^{T_0} -u(c_t^j).$$

(5.21) implies that there are infinitely many j, such that $A + (1/j) \geqslant A_1 > A$. This is a contradiction. Hence, $B = A$.

Now, it can be shown that $\langle k^*, r^*, y^*, c^* \rangle$ is an optimal program. Consider any feasible program $\langle k, r, y, c \rangle$. By Lemma 5.3, it is either good or bad. If it is bad, then

(5.22) $$\limsup_{T \to \infty} \sum_{t=1}^{T} [u(c_t) - u(c_t^*)] = -\infty$$

so that (2.4) is clearly satisfied. If it is good, then $\Sigma_{t=1}^{\infty} u(c_t)$ is convergent, so that

(5.23) $$\limsup_{T \to \infty} \sum_{t=1}^{T} [u(c_t) - u(c_t^*)] = \sum_{t=1}^{\infty} u(c_t) - \sum_{t=1}^{\infty} u(c_t^*) \leqslant A - A$$

by definition of A and construction of $\langle k^*, r^*, y^*, c^* \rangle$. Hence (2.4) is again satisfied, This proves that $\langle k^*, r^*, y^*, c^* \rangle$ is optimal.

REMARK: The proof of Theorem 5.1 follows closely the method of Brock [**1**, p. 278]. Thus the method of proving the existence of an optimal program in traditional growth models (in which exhaustible resources are not included as essential factors of production) suffices in the present model too, once it is shown that a good program exists.

The condition (5.13) may appear somewhat complicated, but it is quite simple to apply to cases, where the utility and production functions are given by well-known parametric forms, as the following example shows.

EXAMPLE 5.1: Consider a case where $G(k, r) = k^{\hat{\alpha}} r^{\hat{\beta}}$ ($\hat{\alpha}$, $\hat{\beta}$ are positive constants, $\hat{\alpha} + \hat{\beta} = 1$), and $u(c) = -c^{1-\sigma}$ ($\sigma > 1$). Then Assumptions 1–8 are satisfied.

Suppose, now, that $\hat{\alpha} > \hat{\beta}$ and $\sigma > [(1-\hat{\beta})/(\hat{\alpha} - \hat{\beta})]$. Then there is α^* such that $0 < \alpha^* < \hat{\alpha}$ and $\sigma > [(1-\beta^*)/(\alpha^* - \beta^*)]$, i.e., $(\sigma - 1)\delta^* > 1$. So,

$$\int_1^\infty w(v; \alpha^*)\, dv = \int_1^\infty [1/v^{(\sigma-1)\delta^*}]\, dv < \infty.$$

Hence, (5.12) and (5.13) are satisfied, and an optimal program exists. This is a result proved by Solow [**18**] and Dasgupta and Heal [**5**].

6. A NON-EXISTENCE RESULT

It is worthwhile to show that the existence theorem of the previous section was not obtained under overly strong sufficient conditions. This is best demonstrated by proving that if an optimal program exists, then conditions "close to" (5.12) and (5.13) must be satisfied. However, before doing that, it is worthwhile to justify the use of Assumption 8 in Theorem 5.1, as follows:

LEMMA 6.1: *Under Assumptions* 1–7, *if an optimal program exists, then*

$$\sup_{c \geq 0} u(c) < \infty. \tag{6.1}$$

PROOF: Suppose an optimal program exists; call it $\langle k^*, r^*, y^*, c^* \rangle$. Then by Theorem 3.1, there exists a sequence $\langle p^*, q^* \rangle$ of nonnegative prices satisfying (2.5), (2.6), (2.7), and (2.8).

Using (2.5) and (2.6), we have for $t \geq 1$,

$$\begin{aligned} u(c_{t+1}^*) - u(c_t^*) &\leq p_t^*(c_{t+1}^* - c_t^*) \\ &= p_t^*(y_{t+1}^* - k_{t+1}^*) - p_t^*(y_t^* - k_t^*) \\ &= [p_t^* y_{t+1}^* - p_{t-1}^* k_t^* - q_{t-1}^* r_t^*] + [p_{t-1}^* k_t^* - p_t^* k_{t+1}^*] \\ &\quad + q_{t-1}^* r_t^* - [p_t^* y_t^* - p_{t-1}^* k_{t-1}^* - q_{t-1}^* r_{t-1}^*] \\ &\quad - [p_{t-1}^* k_{t-1}^* - p_t^* k_t^*] - q_{t-1}^* r_{t-1}^* \\ &\leq [p_{t-1}^* k_t^* - p_t^* k_{t+1}^*] - [p_{t-1}^* k_{t-1}^* - p_t^* k_t^*] \\ &\quad + q_{t-1}^*(r_t^* - r_{t-1}^*). \end{aligned}$$

Then, using (2.7) for $T \geqslant 1$, we have

$$\sum_{t=1}^{T} [u(c_{t+1}^*) - u(c_t^*)] \leqslant [p_0^* k_1^* - p_T^* k_{T+1}^*] - [p_0^* k_0^* - p_T^* k_T^*] + q_0^* \sum_{t=1}^{T} (r_t^* - r_{t-1}^*)$$
$$\leqslant p_0^* k_1^* + p_T^*(k_T^* - k_{T+1}^*) + q_0^* \underset{\sim}{S}.$$

By Theorem 4.1, $k_{T+1}^* > k_T^*$ for $T \geqslant 0$, so

$$\sum_{t=1}^{T} [u(c_{t+1}^*) - u(c_t^*)] \leqslant p_0^* k_1^* + q_0^* \underset{\sim}{S}. \tag{6.2}$$

The right-hand side of (6.2) is independent of T; denote it by M. Then $0 < M < \infty$ and (6.2) implies that for $T \geqslant 1$,

$$u(c_{T+1}^*) \leqslant u(c_1^*) + M.$$

Denote $u(c_1^*) + M$ by $\bar{M}$. Then (6.3) implies that $u(c_t^*) \leqslant \bar{M}$ for $t \leqslant 1$. By Theorem 4.1, $c_t^* \to \infty$ as $t \to \infty$. Hence, u is bounded above, proving (6.1).

Given Lemma 6.1, the rest of the analysis of this section will be confined to the case where (6.1) holds, that is, where Assumption 8 holds. Then, as in Section 5, we shall normalize $\bar{u} = \sup_{c \geqslant 0} u(c)$ to zero.

Following Brock and Gale [**2**, pp. 234–235], I define the *elasticity of the utility function by*

$$d(c) = -[u'(c)c]/u(c) \qquad \text{for} \quad c > 0. \tag{6.4}$$

The *upper and lower asymptotic elasticities of the utility function* ($\bar{d}$ and $\underline{d}$, respectively) are defined by

$$\bar{d} = \limsup_{c \to \infty} d(c); \qquad \underline{d} = \liminf_{c \to \infty} d(c). \tag{6.5}$$

I will now make an additional assumption:

ASSUMPTION 9: $\underline{d} > 0$.

The following result establishes additional necessary conditions for an optimal program to exist (that is, in addition to (6.1)).

THEOREM 6.1: *Under Assumptions* 1–9, *if an optimal program exists, then*

$$\bar{\alpha} \geqslant \bar{\beta} \quad \textit{and} \tag{6.6}$$

$$\int_1^{\infty} w(v; \alpha)\, dv < \infty \qquad \textit{for all } \alpha > \bar{\alpha}. \tag{6.7}$$

PROOF: Suppose an optimal program exists; call it $\langle k, r, y, c \rangle$. By Corollary 3.2 and Theorem 4.1, $c_t > 0$ and $c_{t+1} > c_t$ for $t \geqslant 1$, so that $\inf_{t \geqslant 1} c_t = c_1 > 0$. Also, by Theorem 4.1, $k_t \to \infty$ and $r_t \to \infty$, as $t \to \infty$, so that $x_t \to \infty$. Suppose, now, that (6.6) is violated, i.e., $\bar{\alpha} < \beta$. Then, choose $\alpha > \bar{\alpha}$ such that $\alpha < \beta$. Then there is $\underline{x} < \infty$ such that $G(k, r) = r f(x) \leqslant r x^{\alpha} = k^{\alpha} r^{\beta}$ for $x \geqslant \underline{x}$. Choose $\tau < \infty$ such that $x_t \geqslant \underline{x}$ for $t \geqslant \tau$. Then $G(k_t, r_t) \leqslant k_t^{\alpha} r_t^{\beta}$ for $t \geqslant \tau$. Then, for $t \geqslant \tau$, $k_{t+1} - k_t \leqslant G(k_t, r_t) \leqslant k_t^{\alpha} r_t^{\beta}$. By Theorem 4.1, $k_{t+1} \geqslant k_t$ for $t \geqslant 0$, so for $t \geqslant \tau$

$$(6.8) \qquad [k_{t+1}^{\beta} - k_t^{\beta}] \leqslant [(k_{t+1} - k_t)/k_t^{\alpha}] \leqslant r_t^{\beta}.$$

Summing the inequality in (6.8) from $t = \tau$ to $t = T > \tau$, and using Holder's inequality,

$$(6.9) \qquad [k_{T+1}^{\beta} - k_{\tau}^{\beta}] \leqslant \sum_{t=\tau}^{T} r_t^{\beta} \leqslant \left[\sum_{t=\tau}^{T} r_t\right]^{\beta} [T - \tau + 1]^{\alpha}.$$

Hence for $t \geqslant \tau$ we certainly have $k_{t+1}^{\beta} \leqslant \underset{\sim}{S}^{\beta}(t+1)^{\alpha} + k_{\tau}^{\beta}$. Thus, there is $H < \infty$, such that for $t \geqslant \tau$,

$$(6.10) \qquad k_t \leqslant H t^{\gamma}.$$

Now, for $t \geqslant \tau$, $k_t^{\alpha} r_t^{\beta} \geqslant G(k_t, r_t) = (k_{t+1} - k_t) + c_{t+1} \geqslant c_{t+1}$ (since $k_{t+1} \geqslant k_t$ for $t \geqslant 0$ by Theorem 4.1) $\geqslant c_1$ (since $c_{t+1} \geqslant c_t$ for $t \geqslant 1$ by Theorem 4.1). Using (6.10),

$$(6.11) \qquad r_t^{\beta} \geqslant \frac{c_1}{k_t^{\alpha}} \geqslant \frac{c_1}{H t^{\alpha\gamma}}.$$

Using (6.11), we have for $t \geqslant \tau$, $r_t \geqslant [c_1/H]^{(1/\beta)}/[t^{(\alpha^2/\beta^2)}]$. But $\alpha < \beta$, so $\Sigma_{t=\tau}^{\infty} r_t$ is divergent, a contradiction. Hence (6.6) must be satisfied.

To establish (6.7), we note first that for $t \geqslant 1$, $[-u(c_t)] = [u'(c_t)c_t/d(c_t)]$. By Theorem 4.1, $c_t \to \infty$ as $t \to \infty$. So $d(c_t) \geqslant [\underline{d}/2] > 0$ for t large. Hence there is $D > 0$ such that $D \leqslant d(c_t)$ for $t \geqslant 1$. Thus, for $t \geqslant 1$,

$$(6.12) \qquad [-u(c_t)] \leqslant [(1/D)u'(c_t)c_t] = (1/D)p_t c_t.$$

Using (6.12) and noting from Corollary 3.1 that $\Sigma_{t=1}^{\infty} p_t c_t < \infty$, we have

$$(6.13) \qquad \sum_{t=1}^{\infty} [-u(c_t)] < \infty.$$

Pick any $\alpha > \bar{\alpha}$, and note that by (6.6) $\alpha > \beta$. Then there is $\bar{x} < \infty$ such that, for $x \geqslant \bar{x}$, $f(x) \leqslant x^{\alpha}$. By Theorem 4.1 we know that $x_t \to \infty$ as $t \to \infty$. Hence there is $t_1 < \infty$ such that, for $t \geqslant t_1$, $G(k_t, r_t) = r_t f(x_t) \leqslant k_t^{\alpha} r_t^{\beta}$. Thus, for $t \geqslant t_1$, $k_{t+1} - k_t \leqslant G(k_t, r_t) \leqslant k_t^{\alpha} r_t^{\beta}$. So for $t \geqslant t_1$,

$$(6.14) \qquad [k_{t+1}^{\beta} - k_t^{\beta}] \leqslant [(k_{t+1} - k_t)/k_t^{\alpha}] \leqslant r_t^{\beta}$$

using the fact from Theorem 4.1 that $k_{t+1} \geqslant k_t$ for $t \geqslant 0$. Now, summing the inequality in (6.14) from $t = t_1$ to $t = t_2 > t_1$, and using Holder's inequality, we have

$$(6.15) \qquad [k_{t_2+1}^{\beta} - k_{t_1}^{\beta}] \leqslant \sum_{t=t_1}^{t_2} r_t^{\beta} \leqslant \left[\sum_{t=t_1}^{t_2} r_t\right]^{\beta} [t_2 - t_1 + 1]^{\alpha}.$$

Hence for $t \geq t_1$ we certainly have $k_{t+1}^{\beta} \leq \underset{\sim}{S}^{\beta}(t+1)^{\alpha} + k_{t_1}^{\beta}$. Thus, there is $K_0 < \infty$ such that, for $t \geq 0$,

$$(6.16) \quad k_t \leq K_0(t+1)^{\gamma}.$$

Note that, for $t \geq t_1$, $G(k_t, r_t) \leq k_t^{\alpha} r_t^{\beta}$, so that we have $[G(k_t, r_t)/k_t^{\varepsilon}] \leq k_t^{\alpha-\varepsilon} r_t^{\beta} = [k_t z_t^{\alpha}]^{(\beta/\alpha)}$. Also, $G_{r_t} \leq [G(k_t, r_t)/r_t] = f(x_t) \leq x_t^{\alpha}$, so that $[k_t/G_{r_t}] \geq (k_t/x_t^{\alpha}) \geq k_t z_t^{\alpha}$. Using Corollary 3.2, we know that $(k_t/G_{r_t}) \to 0$ as $t \to \infty$. Hence there is $K_1 < \infty$ such that $[G(k_t, r_t)/k_t^{\varepsilon}] \leq K_1$ for $t \geq 0$. Noting that $c_{t+1} \leq G(k_t, r_t)$ for $t \geq 0$ (since $k_{t+1} \geq k_t$ for $t \geq 0$, by Theorem 4.1), and using (6.16) we finally have

$$(6.17) \quad c_{t+1} \leq [G(k_t, r_t)/k_t^{\varepsilon}]k_t^{\varepsilon} \leq K_1 k_t^{\varepsilon} \leq K_1 H_0^{\varepsilon}(t+1)^{\varepsilon\gamma}.$$

Hence there is $K_2 < \infty$ such that for $t \geq 1$, $c_t \leq K_2 t^{\delta}$, by using (6.17). Using this information in (6.13), we have

$$\sum_{t=1}^{\infty} w[K_2^{(1/\delta)}t; \alpha] = \sum_{t=1}^{\infty} [-u(K_2\, t^{\delta})] \leq \sum_{t=1}^{\infty} [-u(c_t)] < \infty.$$

Hence, by the Maclaurin-Cauchy integral test we have

$$(6.19) \quad \int_1^{\infty} w(v; \alpha)\, dv < \infty.$$

Since $\alpha > \bar{\alpha}$ was arbitrarily chosen, the theorem is proved.

The following nonexistence result is easily obtained from Theorem 6.1. It is, therefore, stated without proof.

Corollary 6.1: *Under Assumptions* 1–9 *there does not exist an optimal program if either*

$$(6.20) \quad \bar{\alpha} < \bar{\beta}$$

or, for some $\alpha > \bar{\alpha}$,

$$(6.21) \quad \int_1^{K} w(v; \alpha)\, dv \to \infty, \qquad as \quad K \to \infty.$$

The following example shows that this nonexistence result can be easily applied to cases where the utility and production functions are given by well-known parametric forms.

Example 6.1: Let $G(k, r) = k^{\hat{\alpha}} r^{\hat{\beta}}$ (where $\hat{\alpha}$, $\hat{\beta}$ are positive constants, $\hat{\alpha} + \hat{\beta} = 1$) and $u(c) = -c^{1-\sigma}$ (where $\sigma > 1$). Notice that Assumptions 1–9 are satisfied. Hence, if $\hat{\alpha} < \hat{\beta}$, no optimal program exists. If $\hat{\alpha} > \hat{\beta}$, and $\sigma < [(1-\hat{\beta})/(\hat{\alpha} - \hat{\beta})]$, then there is some $\alpha > \hat{\alpha}$ such that $\sigma < [(1-\beta)/(\alpha-\beta)]$; that is, $(\sigma-1)\delta < 1$.

$$\lim_{K\to\infty} \int_1^K w(v;\alpha)\,dv = \lim_{K\to\infty} \int_1^K [1/v^{(\sigma-1)\delta}]\,dv = \infty.$$

So (6.21) is satisfied and no optimal program exists.

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